Fusion Performance Analysis with the Correlation

Xianghui Yuan Chongzhao Han and Feng Lian
MOE Key Lab for Intelligent and Networked Systems
State Key Lab for Manufacturing Systems Engineering
Xi'an Jiaotong University
Xi'an, Shaanxi, P.R.China.
{xhyuan,czhan, lianfeng}@mail.xjtu.edu.cn

Abstract - *Estimation fusion performance with the effect of correlation is analyzed. First, the correlation type is classified into two categories: same source correlation and different source correlation. Second, with the best linear unbiased estimation (BLUE) criteria, the scalar case is analyzed with the effect of correlation. For the same source correlation, with the same absolute value, the fusion result for negative correlation is better than positive correlation. For the different source correlation, with independent extra information, stronger correlation can lead to better fusion result. Finally, the conclusions are supported by some examples.*

Keywords: Estimation fusion, Performance analysis, Correlation coefficient

1 Introduction

Estimation fusion is the problem of how to best utilize useful information contained in multiple sets of data for the purpose of estimating an unknown quantity – a parameter or process [1]. These data are usually from multiple sources, e.g. multiple sensors, multiple estimators. If sensor can be regarded as some kind of estimator, all the data can be seen as provided by the estimators, which are usually correlated.

For example, when different sensors measure a same object in a noisy environment, the measurements are usually correlated. For another example, in distributed target tracking system, those tracks are usually correlated due to the same initial data, common process noise, etc. Because the data are the estimation for the same object, the above kind of correlation can be called "same source correlation".

There are some other kind of correlation. For example, Constant velocity (CV) model can provide position and velocity's estimation and the two estimations are correlated because of the kinematic model. For another example, target's range, azimuth and elevation can be measured by radar. Even the radar's original measurements are uncorrelated, they will be correlated when they are transformed from spherical coordinate to Cartesian coordinate. Because the correlated data stand for different objects' estimation, this kind of correlation can be called "different source correlation".

The purpose of this paper is to show the effect of the correlation to the fusion result. The rest of the paper is organized as follows: problem formulation and fusion algorithm is shown in section 2. Performance analysis is given in section 3 and some examples are given in section 4. Section 5 is the conclusion.

2 Problem formulation and fusion algorithm

2.1 Same source correlation

Only the scalar case is considered here. Assume the estimator $\hat{x}_1, \hat{x}_2$ satisfy the following equation:

$$\hat{x}_1 = x + v_1 , \quad \hat{x}_2 = x + v_2 \tag{1}$$

where x is the true estimatee and v_1, v_2 are the estimation errors. Assume the estimation errors have the zero mean :

$$E[v_1] = E[v_2] = 0 \tag{2}$$

The followings are the variances and the covariance of the estimation errors :

$$\operatorname{var}(v_1) = \sigma_1^2 , \operatorname{var}(v_2) = \sigma_2^2 \tag{3}$$

$$\operatorname{cov}(v_1, v_2) = \rho\sigma_1\sigma_2 , \rho \in [-1,1] \tag{4}$$

where ρ is the correlation coefficient. Because $\hat{x}_1, \hat{x}_2$ are both the estimation for same x, here ρ stands for the same source correlation coefficient.

If the two estimators are fused to obtain a new estimation, it becomes a standard track fusion problem. With the BLUE criteria [2], the fusion equation will be :

$$\hat{x} = \bar{x} + P_{xz} P_{zz}^{-1} (z - \bar{z}) \tag{5}$$

$$P = P_{xx} - P_{xz} P_{zz}^{-1} P_{zx} \tag{6}$$

where $\bar{x}$ is the prior mean and z is the measurement. Just like [3], denote x_1 as the prior mean and x_2 as the measurement, the covariance term of (5) and (6)will become :

$$P_{xz} = \sigma_1^2 - \rho\sigma_1\sigma_2 \tag{7}$$

$$P_{zz} = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \quad (8)$$

With (7)and (8), the final fusion equation becomes :

$$\hat{x} = \frac{(\sigma_2^2 - \rho\sigma_1\sigma_2)\hat{x}_1 + (\sigma_1^2 - \rho\sigma_1\sigma_2)\hat{x}_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \quad (9)$$

The corresponding variance becomes :

$$\sigma^2 = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \quad (10)$$

2.2 Different source correlation

The following problem is considered: there are two estimators, one can provide the complete estimation to one estimatee, the other can only provide part of the estimation to the estimatee. Assume X is the estimatee, which can be expressed as $X = [x \quad y]^T$.

Estimator 1:

$$\hat{X}_1 = \begin{bmatrix} \hat{x}_1 \\ \hat{y}_1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} v_{x_1} \\ v_{y_1} \end{bmatrix} \quad (11)$$

Estimator 2:

$$\hat{X}_2 = \hat{x}_2 = x + v_{x_2} \quad (12)$$

It can be seen that x is the common part, y is the extra part. x and y are scalars here.

The mean, covariance and cross covariance of the noise are:

$$E\left[v_{x_1}\right] = E\left[v_{y_1}\right] = E\left[v_{x_2}\right] = 0 \quad (13)$$

$$\operatorname{cov}\left(\begin{bmatrix} v_{x_1} \\ v_{y_1} \end{bmatrix}\right) = \begin{bmatrix} \sigma_{x_1}^2 & \rho_{x_1y_1}\sigma_{x_1}\sigma_{y_1} \\ \rho_{x_1y_1}\sigma_{x_1}\sigma_{y_1} & \sigma_{y_1}^2 \end{bmatrix} \quad (14)$$

$$\operatorname{cov}(v_2^x) = \sigma_{x_2}^2 \quad (15)$$

$$\operatorname{cov}\left(\begin{bmatrix} v_{x_1} \\ v_{y_1} \end{bmatrix}, v_{x_2}\right) = \begin{bmatrix} \rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2} \\ \rho_{y_1x_2}\sigma_{y_1}\sigma_{x_2} \end{bmatrix} \quad (16)$$

$\rho_{x_1y_1}$ and $\rho_{y_1x_2}$ stand for the different source correlation coefficient.

To fusion such kind of data, the following three methods can be used:

1) Optimal WLS fuser

2) BLUE fuser without prior

3) Regard one of the data as prior and use BLUE fuser

Although they have different fusion equation, they are actually identical [4]. Here the third method (BLUE fusion with prior) is used.

Regard $\begin{bmatrix} \hat{x}_1 \\ \hat{y}_1 \end{bmatrix}$ as the prior and $\hat{x}_2$ as the measurement, the fusion equation will be:

$$\hat{X} = \bar{X} + K(\hat{x}_2 - \hat{x}_1) = \begin{bmatrix} \hat{x}_1 \\ \hat{y}_1 \end{bmatrix} + \frac{\begin{bmatrix} \sigma_{x_1}^2 - \rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2} \\ \rho_{x_1y_1}\sigma_{x_1}\sigma_{y_1} - \rho_{y_1x_2}\sigma_{y_1}\sigma_{x_2} \end{bmatrix}(\hat{x}_2 - \hat{x}_1)}{\sigma_{x_1}^2 + \sigma_{x_2}^2 - 2\rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2}} \quad (17)$$

$$P = \begin{bmatrix} \sigma_{x_1}^2 & \rho_{x_1y_1}\sigma_{x_1}\sigma_{y_1} \\ \rho_{x_1y_1}\sigma_{x_1}\sigma_{y_1} & \sigma_{y_1}^2 \end{bmatrix} - \frac{\begin{bmatrix} \sigma_{x_1}^2 - \rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2} \\ \rho_{x_1y_1}\sigma_{x_1}\sigma_{y_1} - \rho_{y_1x_2}\sigma_{y_1}\sigma_{x_2} \end{bmatrix}\begin{bmatrix} \sigma_{x_1}^2 - \rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2} \\ \rho_{x_1y_1}\sigma_{x_1}\sigma_{y_1} - \rho_{y_1x_2}\sigma_{y_1}\sigma_{x_2} \end{bmatrix}^T}{\sigma_{x_1}^2 + \sigma_{x_2}^2 - 2\rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2}} \quad (18)$$

x part (common data part) is same as the same source correlation. Here the y part (extra data part) is analyzed. The covariance is used here for performance analysis.

$$\sigma_y^2 = \sigma_{y_1}^2 - \frac{\left(\rho_{x_1y_1}\sigma_{x_1}\sigma_{y_1} - \rho_{y_1x_2}\sigma_{y_1}\sigma_{x_2}\right)^2}{\sigma_{x_1}^2 + \sigma_{x_2}^2 - 2\rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2}} \quad (19)$$

3 Fusion performance analysis

3.1 Same source correlation

From equation(10), the relationship between the fusion performance σ^2 and the correlation coefficient ρ can be analyzed.

It is obvious that only if $\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \neq 0$, equation (10) will be meaningful. So let's discuss from $\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 = 0$.

$$\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 = (\sigma_1 - \sigma_2)^2 + 2(1-\rho)\sigma_1\sigma_2 \quad (20)$$

Since $\sigma_1, \sigma_2 > 0$, only when $\sigma_1 = \sigma_2$ and $\rho = 1$, $\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 = 0$

Case 1: $\sigma_1 = \sigma_2$ and $\rho = 1$

In this case, from (7) and (8), it can be seen that $P_{xz} = P_{zz} = 0$. Since the pseudo inverse of 0 is equal to 0, the fusion result can be got from (5) and (6):

$$x = \hat{x}_1 \tag{21}$$

$$\sigma^2 = \sigma_1^2 \tag{22}$$

It can be physically explained as following : since $\sigma_1 = \sigma_2$ and $\rho = 1$, estimator 2 actually is the same as estimator 1. So estimator 2 can't provide any new information to estimator 1. So the fusion result is the same as estimator 1.

Conclusion 1: when $\sigma_1 = \sigma_2$ and $\rho = 1$, the fused estimator is the same as the original one.

In the other case, $\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \neq 0$, so let's directly use (10) to analyze.

From (10), the following derivation can be got :

$$\frac{d(\sigma^2)}{d\rho} = \frac{2\sigma_1^3\sigma_2^3(\rho - \frac{\sigma_1}{\sigma_2})(\rho - \frac{\sigma_2}{\sigma_1})}{\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)^2} \tag{23}$$

It can be seen that if $\sigma_1 = \sigma_2$ and $\rho \neq 1$,then $\frac{d(\sigma^2)}{d\rho} > 0$.

Case 2: $\sigma_1 = \sigma_2$ and $\rho \neq 1$

In this case, when ρ increases, σ^2 will also increase. Put $\sigma_1 = \sigma_2$ into (10),

$$\sigma^2 = \frac{1}{2}\sigma_1^2(1+\rho) \tag{24}$$

σ^2 is a monotonic increasing linear function of ρ. If $\rho < 0$, $\sigma^2(\rho) < \sigma^2(0)$, which means that negative correlation is better than non-correlation.

When $\rho = -1$, $\sigma^2 = 0$, which means the perfect truth can be obtained. Actually when $\sigma_1 = \sigma_2$ and $\rho = -1$, v_1 and v_2 are always with the opposite sign. The error can be eliminated then the perfect error free fusion result can be got.

When $\rho \to 1$, it can be got from (10) that :

$$\lim_{\substack{\rho \to 1 \\ \sigma_1 = \sigma_2}} \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = \sigma_1 \tag{25}$$

Which is the same as case 1. So case 1 is a special case of case 2.

Conclusion 2: when $\sigma_1 = \sigma_2$, the fused estimator's variance is a monotonic increasing linear function of correlation coefficient. Negative correlation is better than non-correlation and perfect negative correlation can get perfect error free fusion result. Just as mentioned in conclusion 1, perfect positive correlation will be no help for fusion.

Case 3: $\sigma_1 \neq \sigma_2$

Without loss of generality, assume $\sigma_1 > \sigma_2$. From (23), it can be seen that:

$$\frac{d(\sigma^2)}{d\rho} > 0 \qquad \text{if } \rho < \frac{\sigma_2}{\sigma_1} \tag{26}$$

$$\frac{d(\sigma^2)}{d\rho} = 0 \qquad \text{if } \rho = \frac{\sigma_2}{\sigma_1} \tag{27}$$

$$\frac{d(\sigma^2)}{d\rho} < 0 \qquad \text{if } \rho > \frac{\sigma_2}{\sigma_1} \tag{28}$$

From (26)-(28), it can be concluded that when $\rho < \frac{\sigma_2}{\sigma_1}$, σ^2 is a monotonic increasing function of ρ ; when $\rho > \frac{\sigma_2}{\sigma_1}$, σ^2 is a monotonic decreasing function of ρ ; when $\rho = \frac{\sigma_2}{\sigma_1}$, σ^2 will reach its peak maximum value.

Put $\rho = \frac{\sigma_2}{\sigma_1}$ to (10),

$$\sigma^2 = \sigma_2^2 \tag{29}$$

The generalization of the result is the following :

Let $\rho^* = \min(\frac{\sigma_2}{\sigma_1}, \frac{\sigma_1}{\sigma_2})$, when $\rho < \rho^*$, σ^2 is a monotonic increasing function of ρ .When $\rho > \rho^*$, σ^2 is a monotonic decreasing function of ρ .When $\rho = \rho^*$, σ^2 will reach its peak maximum value, which is $\min(\sigma_1^2, \sigma_2^2)$.

When $\rho = \rho^*$, the fusion result is the same as the more precise estimator. It seems that the fusion will be no help in the worse case, but actually the fusion equation can

automatically choose the more precise one's estimator as the fusion result. It also helps in that kind of sense.
From (10), it can be seen that :

$$\lim_{\rho \to 1} \frac{\sigma_1^2 \sigma_2^2 (1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = 0 \quad (30)$$

$$\lim_{\rho \to -1} \frac{\sigma_1^2 \sigma_2^2 (1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = 0 \quad (31)$$

So it can be concluded that, when $\sigma_1 \neq \sigma_2$**,** both perfect postive and negative correlation can obtain perfect fusion result.

Let $\sigma(\rho) < \sigma(0)$,

$$\frac{\sigma_1^2 \sigma_2^2 (1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} < \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad (32)$$

which equals

$$\rho^2(\sigma_1^2 + \sigma_2^2) - 2\rho\sigma_1\sigma_2 > 0 \quad (33)$$

it will get that when $\rho < 0$ or $\rho > \frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$, $\sigma(\rho) < \sigma(0)$.

So when $\rho < 0$ or $\rho > \frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$, correlation is better than non-correlation.

Conclusion 3: when $\sigma_1 \neq \sigma_2$, there exists a certain positive value, which is named ρ^* and $\rho^* = \min(\frac{\sigma_2}{\sigma_1}, \frac{\sigma_1}{\sigma_2})$. When $\rho < \rho^*$, σ^2 is a monotonic increasing function of ρ; when $\rho > \rho^*$, σ^2 is a monotonic decreasing function of ρ; when $\rho = \rho^*$, σ^2 will reach its peak maximum value, which is $\min(\sigma_1^2, \sigma_2^2)$. Both perfect postive and negative correlation can obtain perfect fusion result. when $\rho < 0$ or $\rho > \frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$, correlation is better than non-correlation.

The following compares the negative correlation with the positive correlation and both correlation coefficients have the same absolute value. For some certain $\rho \neq 0$,the following equation can be easily derived from (10):

$$\sigma^2(|\rho|) = \frac{\sigma_1^2 \sigma_2^2 (1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2|\rho|\sigma_1\sigma_2} \quad (34)$$

$$\sigma^2(-|\rho|) = \frac{\sigma_1^2 \sigma_2^2 (1-\rho^2)}{\sigma_1^2 + \sigma_2^2 + 2|\rho|\sigma_1\sigma_2} \quad (35)$$

It can be seen that :

$$\sigma^2(-|\rho|) < \sigma^2(|\rho|) \quad (36)$$

Conclusion 4: With the same absolute value correlation coefficient, the fusion result for negative correlation is better than positive correlation.

3.2 Different source correlation

The analysis for this part from (19) and a simplified scenario is discussed next. Assume estimator 1 and 2 are independent with each other, which means:

$$\operatorname{cov}\left(\begin{bmatrix} v_{x_1} \\ v_{y_1} \end{bmatrix}, v_{x_2}\right) = \begin{bmatrix} \rho_{x_1x_2}\sigma_{x_1}\sigma_{x_2} \\ \rho_{y_1x_2}\sigma_{y_1}\sigma_{x_2} \end{bmatrix} = \mathbf{0} \quad (37)$$

Then (19) becomes:

$$\begin{aligned} \sigma_y^2 &= \sigma_{y_1}^2 - \frac{\left(\rho_{x_1y_1}\sigma_{x_1}\sigma_{y_1}\right)^2}{\sigma_{x_1}^2 + \sigma_{x_2}^2} \\ &= \sigma_{y_1}^2 \left(1 - \frac{\rho_{x_1y_1}^2 \sigma_{x_1}^2}{\sigma_{x_1}^2 + \sigma_{x_2}^2}\right) \end{aligned} \quad (38)$$

$\rho_{x_1y_1}$ is the different source correlation coefficient. The following two conclusions can be derived from (38):

Conclusion 5: if σ_{x_1} , σ_{x_2} and σ_{y_1} are fixed, when $|\rho_{x_1y_1}|$ increases, σ_y^2 will decrease. This means that higher correlation can lead to better fusion result.

Conclusion 6: If σ_{x_1} , σ_{y_1} and $\rho_{x_1y_1}$ are fixed, $\rho_{x_1y_1} \neq 0$, when σ_{x_2} decreases, σ_y^2 will decrease. This means that with the different source correlation, more accurate independent estimator can lead to better fusion result.

4 Examples

4.1 Same source correlation

For the same source correlation, some examples are simulated.
Example 1: Two estimators have the same variances and $\sigma_1^2 = \sigma_2^2 = 9$
Fig.1 shows the relationship between σ^2 and ρ . Conclusion 1 and 2 are supported by Fig.1.

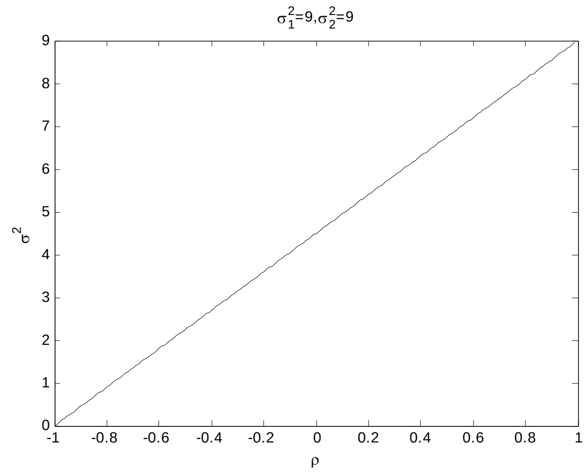

Fig.1 The relationship between σ^2 and ρ when $\sigma_1 = \sigma_2$

Example 2: Two estimators have the different variances and $\sigma_1^2 = 4, \sigma_2^2 = 16$, $\rho^* = 0.5$

Fig.2 is the relationship between σ^2 and ρ . Conclusion 3 and 4 are supported by Fig.2.

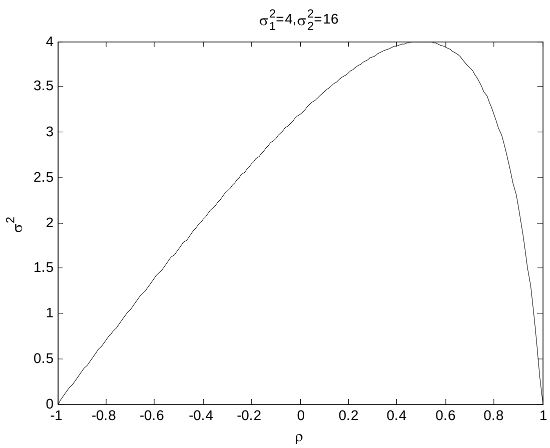

Fig.2 The relationship between σ^2 and ρ when $\sigma_1 \neq \sigma_2$

4.2 Different source correlation

Example 3: One Doppler radar can get range and range rate measurements. The range and range rate measurements are sometimes correlated [5]. Another radar can only get range measurement. The two radars are independent with each other. The state vectors are $\begin{bmatrix} r & \dot{r} \end{bmatrix}'$ and r , respectively. The corresponding covariances are:

$$P_1 = \begin{bmatrix} \sigma_{r1}^2 & \rho\sigma_{r1}\sigma_{\dot{r}} \\ \rho\sigma_{r1}\sigma_{\dot{r}} & \sigma_{\dot{r}}^2 \end{bmatrix}, \ P_2 = \sigma_{r2}^2$$

After fusion,

$$P_{\dot{r}} = \sigma_{\dot{r}}^2 \left(1 - \frac{\rho^2 \sigma_{r1}^2}{\sigma_{r1}^2 + \sigma_{r2}^2} \right) \qquad (39)$$

Let $P_1 = \begin{bmatrix} 10 & 10\rho \\ 10\rho & 10 \end{bmatrix}, P_2 = 10$,

Fig.3 shows $P_{\dot{r}}$ as a function of $|\rho|$, which changes from 0 to 1.Conclusion 5 are supported by Fig.3.

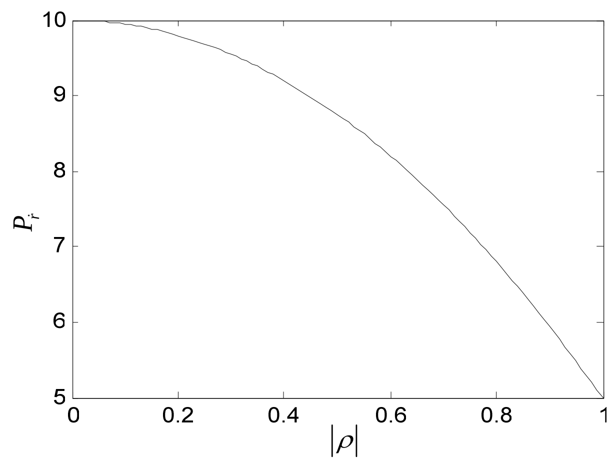

Fig.3 $P_{\dot{r}}$ as a function of $|\rho|$

Example 4: The scenario is the same as example 3 and the correlation coefficient is fixed.

Let $P_1 = \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}, \ P_2 = \sigma_{r2}^2$

Fig.4 shows $P_{\dot{r}}$ as a function of σ_{r2}^2 ,which changing from 0 to 10 . Conclusion 6 is supported by Fig.4.

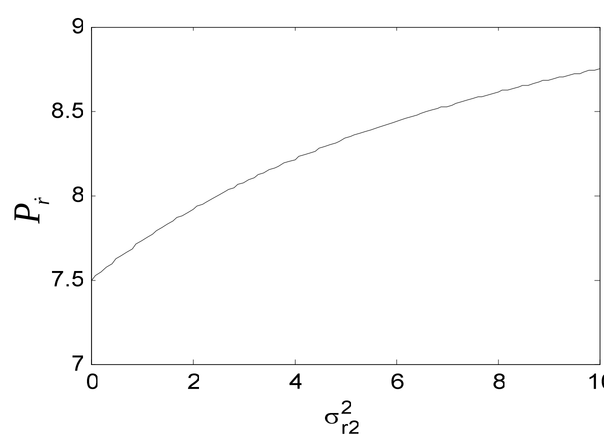

Fig.4 The relationship between radar's range accuracy and fused range rate accuracy

5 Conclusion

The correlation type is classified into two categories: same source correlation and different source correlation. With the best linear unbiased estimation (BLUE) criteria, fusion performance with the scalar case is analyzed. For the same source correlation, negative correlation is better than positive correlation. For the different source correlation, with the independent extra information, stronger correlation's fusion result is better than the weak correlation.

6 Acknowledgement

This work is supported by the Grant for State Key Program for Basic Research of China (973)(No.2007CB311006), Natural Science Foundation of China(No. 60805016).

References

[1] Li, X.R., Zhu, Y.M., Wang, J. and Han, C.Z. *Optimal linear estimation fusion: Part .I. Unified fusion rules.* IEEE Transactions on Information Theory, Vol.49 , No.9, Sep 2003: 2192-2208.

[2] Bar-shalom, Y， Li, X.R. and Kirubarajan T. *Estimation with applications to tracking and navigation: Theory, Algorithms and Software*, Wiley,New York, 2001, ch.3.

[3] Yaakov Bar-shalom and Leon Campo. *The effect of the common process noise on the two-sensor fused-track covariance.* IEEE Transactions on Aerospace and Electric Systems, Vol 22, No.6, pp803-805, Nov 1986.

[4] Li, X.R., Zhang, K.S., Zhao, J. and Zhu, Y.M. *Optimal linear estimation fusion. Part V. Relationships.* Proceedings of the Fifth International Conference on Information Fusion, Annapolis, Maryland, USA July 2002,497-504

[5] Bar-Shalom Y. *Negative correlation and optimal tracking with Doppler measurements*. IEEE Transactions on Aerospace and Electronic Systems, 2001, 37 (3): 1117-1120.